Investigation of Nonverbal Proof Skills of Preservice Mathematics Teachers': A Case Study

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ABSTRACT

The importance of both proof and visualization has been frequently emphasized in mathematics education. Visual proof or nonverbal proofs are defined as diagrams or illustrations that help us to see why a mathematical expression is correct, and how to begin to prove the accuracy of this statement. The aim of this research is to examine nonverbal proof skills of preservice mathematics teachers. The study was carried out with case studies, one of the qualitative research designs. The participants of the study consisted of 53 preservice mathematics teachers at a state university in Central Anatolia, Turkey. The data were collected with a sample of three nonverbal proof samples directed to preservice teachers. The analysis of the data classified the preservice teachers' responses according to their similarities and differences. The findings showed that preservice teachers generally associate images with geometric figures. In addition, it was also seen that those who saw the visual relationship between the given visual and mathematical expression used it to show the expression as correct instead of proofing the visual.

Key Words: Proof, Nonverbal poof, Visual proof

DOI Number: http://dx.doi.org/10.22521/jesr.2019.91.2

Received Date: 05.03.2019 Accepted Date: 15.04.2019

Atıf için / Please cite as:

Demircioğlu, H. (2019). Investigation of nonverbal proof skills of preservice mathematics teachers': A case study. *Eğitim Bilimleri Araştırmaları Dergisi - Journal of Educational Sciences Research, 9*(1), 21-39. https://dergipark.org.tr/ebader

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INTRODUCTION

The importance of both proof and visualization has been frequently emphasized in the learning and teaching of mathematics. A proof is used to show whether or not an assumption is correct (Heinze, & Reiss, 2004) and is at the core of both mathematics and mathematics education (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002a, 2000b). Nevertheless, it has always been a difficult issue (Jones, 2000). Although it is a challenging subject or skill, one of the alternative methods that can be applied to acquire the skill of proof is gaining the ability of proof by visualization; that is, the use of nonverbal proofs. The nonverbal proof is one of the methods that supports both making proof and proof skill. Alsina and Nelsen (2010) described nonverbal proofs as "visual arguments." A picture is better than a thousand words (Rösken, & Rolka, 2006; Thornton, 2001). As Casselman (2000) stated, proofs are common in many cultures. In fact, proof with diagrams-pictures has a long history (Foo, Pagnucco, & Nayak, 1999). Nonverbal proof studies have been published in various journals since the mid-1970. Especially in recent years, nonverbal proofs have started to form part of mathematics education (Polat, & Demircioğlu, 2016), and this interest has been increasing rapidly in both mathematics research and mathematics education practices (Bardelle, 2009).

Nonverbal proof is a mathematical drawing describing the proof of a mathematical expression, but without proving the formal argument with words (Bell, 2011). Bardelle (2009) defined visual (nonverbal) proofs as deductive steps based on figures, diagrams, and graphs. According to Alsina and Nelsen (2010), they are pictures or diagrams that help to see *why* a certain mathematical expression can be correct and also to see *how* that expression can begin to prove it. According to Bardelle (2009), visual proofs are those which are not presented with any comments in verbal language (that is nonverbal), but are only based on diagrams, maybe numbers, letters, arrows, dots, and symbolic expressions associated with each other; the proof that is, or configuration of the proof, is left to the reader. The nonverbal proof uses visual representation, meaning that pictures and visuals are used to represent a mathematical equation, theorem or idea (Casselman, 2000; Gierdien, 2007). In summary, the nonverbal proof is also used to acquire the ability to make a proof or to explain the proof process better in addition to demonstrating why a proof is correct.

Nonverbal proofs play an important role in mathematics classes from primary education through to university level (Alsina, & Nelsen, 2010). In general, there are many examples in proofs of many theorems in many fields such as algebra, analysis, trigonometry and more specifically, "Total of cubes" or the "Nicomachus theorem" (Stucky, 2015), and "Alternate series test" by using the fields of rectangles (Hammack, & Lyons, 2006). It can also be used in the history of mathematics courses (Bell, 2011). There are indeed examples of verbal evidence with almost every subject, and these examples can be found on various websites (e.g., www.illuminations.nctm.org, www.cut-the-knot.org, in two books written by Nelsen (1993, 2000), as well as in journal articles (e.g., Bell, 2001; Gierdien, 2007). Some nonverbal proofs can also be found on interactive sites such as nonverbal proof of Pythagoras (Bell, 2011); see http://illuminations.nctm.org/ActivityDetail.aspx?ID=30. Stucky (2015) stated that nonverbal proofs are a useful pedagogical tool. Nonverbal proofs are more interesting and acceptable for students than classical proofs (Štrausová, & Hašek, 2013), effective in understanding students' process of proof (Bell, 2011), and can play an important role in the process of understanding various mathematical features (Štrausová, & Hašek, 2013). Miller

(2012) stated that nonverbal proofs are a valuable tool in mathematics, especially in the teaching of mathematics. To support this claim, he expressed that a student who can show the accuracy of the formula of first n integer sum by the inductive method cannot be convinced why this is true. Here, he explained that nonverbal proofs can be effective. Hammack and Lyons (2006) stated that in using visual proof for the convergence of Alternate series, student success increased and the generated proofs were clearer than the proofs found in the standard books. He explained that by using the area of rectangles, concreteness and abstraction replaced and eliminated some of the difficulties.

Although there are many nonverbal proofs to be found in the literature, it can be observed that there have not been many studies about the effects of nonverbal proofs on teaching and proofing skills. The studies carried out generally show the theoretical structure, history and examples of nonverbal proofs (Alsina, & Nelsen, 2010; Miller, 2012) in the classroom with application examples (Bell, 2011; Gierdien, 2007), or student difficulties in nonverbal proof practices (Bardelle, 2009), approaches of talented secondary school students' making proof (Uğurel, Moralı, & Karahan, 2011), and the approaches of preservice primary school teachers about making proof with concrete models (Doruk, Kıymaz, & Horzum, 2012). In their studies, Demircioğlu and Polat (2015, 2016) stated that preservice teachers expressing nonverbal proofs were effective in gaining problem-solving, understanding, mental, reasoning, generalization, processing, analysis and thinking skills. At the same time, it has been seen that the most difficult aspects of nonverbal proofs are being unable to understand given figures, the lack of explanation, inability to establish a relationship between a nonverbal proof and an algebraic proof, lack of field knowledge, and lack of adequate sources. Indeed, since only one visual is used in nonverbal proofs, it is an important skill to be able to recognize when seeing it, to make a proof using only the visual information, and to make comments. Therefore, the current study aims to examine both the nonverbal proof and visualization skills of mathematics teachers.

METHOD

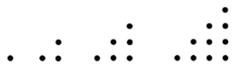
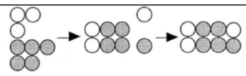
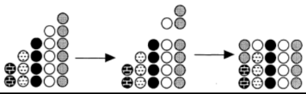
Research Model

This study employed case study methodology to investigate preservice secondary mathematics teachers' nonverbal proof skills. Case study research is a qualitative approach in which a bounded case is explored over time, through detailed, in-depth data collection involving multiple sources of information (Yıldırım, & Şimşek, 2005; Yin, 2003). The participants of the study consisted of 53 preservice teachers in the final year of secondary mathematics undergraduate education at a state university in Central Anatolia, Turkey. 32 female and 21 male preservice mathematics teachers participated in the current research. Participants were identified by convenience sampling method. All of the preservice teachers who took part in the study had received field and field education courses given at the bachelor's level as they were in their final year. For this reason, it was accepted that they had the necessary conceptual and procedural knowledge about the concepts contained in the nonverbal proofs and that they had already gained the ability of proofing. The fact that these preservice teachers would be practicing teachers in their near future made the studying of these skills particularly meaningful.

Data Collection Tool

Three questions were constructed in order to examine the nonverbal proof skills of the participant preservice teachers. The first of these questions is an example of modeling the sum of numbers from 1 to n, which are included in many sources. The second and third questions were obtained from Miyazaki (2000). The second question relates to "the sum of two odd numbers is even" and the third question is similar to the first, but can be used in the proof of different expressions.

Table 1. *Data collection tool questions*

Question	Image	Text
1		Explain what it means to you
2		Explain what it means to you
3		Explain what it means to you

Although the first and third questions appear to be similar, the first question is a visual that can reveal the generalization process and would be familiar to the preservice teachers. However, the third question is accepted as an example of specialization skill and includes a process of making a provision about the general rule based on similar examples. Therefore, the findings of these two problems are important for comparative purposes. The second question bears similarity with the third question. This example is similar to the visuals used when modeling skill is being examined, but here specialization skill has been examined. It provides pieces of evidence about whether or not a general rule can be seen from specific examples. In this sense, it is important to compare the findings of the second and third questions.

Data Collection Procedure and Data Analysis

Data were collected in writing. Each participant was assigned a random number. Data were then transferred to a computer environment. The participants' responses were then grouped according to their similarities, and content analysis was subsequently performed. Content analysis is the access to concepts and relationships that can explain the collected data. In content analysis, the fundamental process is to bring together similar data in the context of specific concepts and themes and to interpret them in a way that the reader can understand (Özdemir, 2010; Yıldırım, & Şimşek, 2005).

In this way, the category and subcategories were obtained so as to reveal what the preservice teachers understood from the given visuals and to examine their nonverbal proof skills. In the process of obtaining the categories and subcategories, the expressions included in the answers given were coded directly. In other words, "Sum of numbers from 1 to n…," "Triangle," etc. were coded and a frequency table created. In the following stage, each category was examined and subcategories were formed. These categories and subcategories were examined separately by two different experts in the field. The researcher and the independent reviewer discussed each of these categories and subcategories. As a consequence of the discussion, the categories were finalized. Direct quotations of the participants' statements have been used within the reporting of the findings in order add to the reliability of the data.

FINDINGS

Three research questions directed this study in the examination of nonverbal proof skills of preservice mathematics teachers. The findings obtained from each question are first evaluated separately and then together.

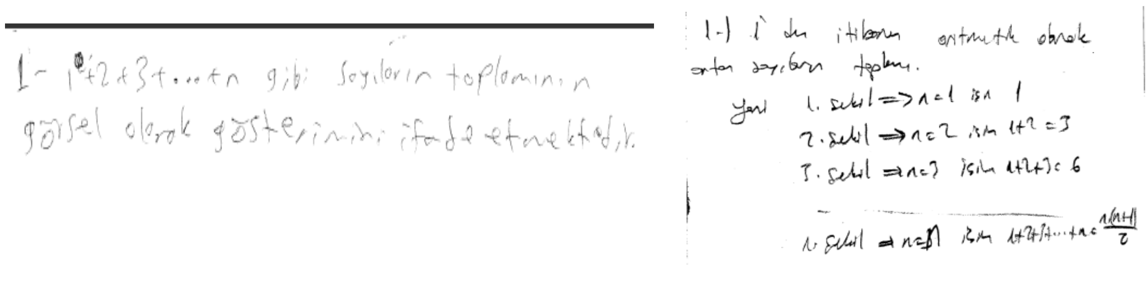
Findings from the First Question

The first question directed to the preservice teachers was regarding "the sum of numbers from 1 to n," which is usually proofed by way of the induction method in analysis courses. The answers given by the preservice teachers to the given image (see Table 1) are summarized in Table 2.

Table 2. *Preservice teachers' answers to first question*

Category	Subcategory	*f*
Sum of numbers from 1 to n	Sum of numbers from 1 to n and formula Example of modeling the sum of numbers from 1 to n	12
Triangle	Triangle Isosceles triangle Right-angled triangle Isosceles right-angled triangle	14
Pattern	1, 3, 5,…,2n-1 Pattern as figure 1, 3, 6, 10,… and amount of increase 1,2,3,…	13
Pattern and Triangle	Triangle Increasing triangles Nested parallel triangle Isosceles right-angled triangle	5
Other	Pascal's triangle, Right-angled triangle, Euclidean relation Pascal's triangle Binomial expansion Reaching generalizations Geometric series Dots	9

As can be seen from Table 2, a total of 12 preservice teachers participating in the study stated that the given visual was related to the sum of numbers from 1 to n. Four of these preservice teachers explained the visualization of the sum of numbers from 1 to n (see Figure 1a), while eight attempted to verify the number of dots with the formula $\frac{n(n+1)}{2}$ for each unity step (see Figure 1b). However, none of the preservice teachers made a connection between the given visual and the formula $\frac{n(n+1)}{2}$.



a) b)

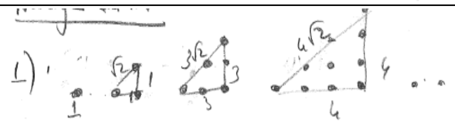
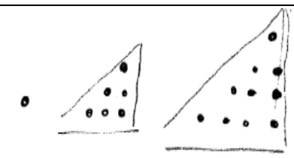
Figure 1. *Preservice teachers' response as "visual related to sum of numbers from 1 to"*

This example has been given in many sources and undergraduate courses. When the answers in this category were examined, it was observed that the preservice teachers were unable to express themselves using the visual $\frac{n\,(n+1)}{2}$, although they stated that the given visual was related to the sum of numbers from 1 to n. It was interpreted as they saw in this example, but they did not perform any activity related to nonverbal proof skills.

Of the preservice teachers, 14 expressed that the given visual was related with "forming a triangle." More specifically, three of the preservice teachers stated that the figure resembled a triangle by combining the dots, three mentioned an isosceles triangle, four stated a right-angled triangle, and four of the preservice teachers mentioned an isosceles right-angled triangle.

As can be seen from Table 3, all three preservice teachers combined the dots around the edges of the figure and expressed the figure as triangles whose edge lengths became longer by one unit for each side (triangles, right-angled triangles, isosceles triangles, isosceles right-angled triangle).

Table 3. *Preservice teachers' responses to "triangle" category*

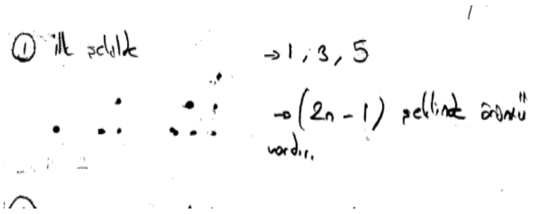
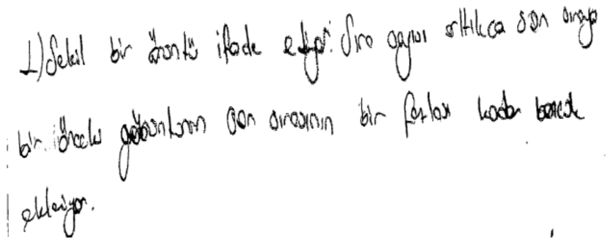
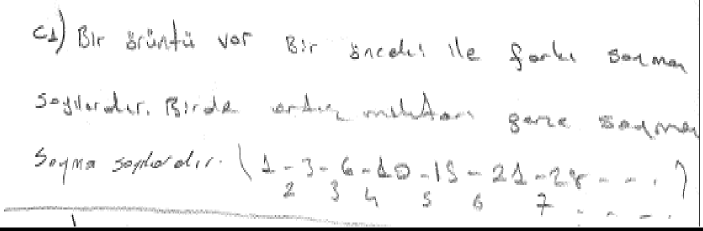
Category	Subcategory	Sample response		*f*
Triangle	Triangle	(1) Tek nokta ile başlayıp ... şeklinde artarak giden üçgen görünümü oluşturan noktalar kümesidir.	3	14
	Isosceles triangle	1)  Tüzlenen üçgen belirtebilirler. Düzenli olarak her satıra nokta eklediğimizde üçgenlerin kenar uzunlukları da türlenen bir birim artar.	3	
	Right-angled triangle	1. şekil ;  bu şekil bana dik üçgeni ifade ediyor.	4	
	Isosceles right-angled triangle	1) Her şeklinde kenar uzunlukları 1 birim artan ikizkenar dik üçgen oluşur.	4	

Here, only one of the three preservice teachers in the "isosceles right-angled triangle" subcategory stated that there was an isosceles triangle, made a connection with dot numbers and wrote the number of points as 1,3,6,10,... This preservice teacher may have inadvertently linked with the triangular numbers. Since this preservice teacher did not mention the pattern, their answer was not evaluated in the pattern category as well.

A total of 13 preservice teachers stated that there was a pattern based on the increase in dots in each step. When Table 4 is examined, it can be seen that the majority of preservice teachers expressed the pattern as 1, 3, 6, 10,… dots and with increasing dots in each step as 1, 2, 3,... In addition, one preservice teacher counted the points incorrectly and thought that after one dot there were three dots; and then generalized this situation by expressing it as 1, 3,... 2n-1.

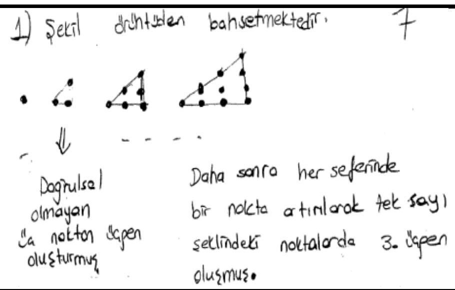
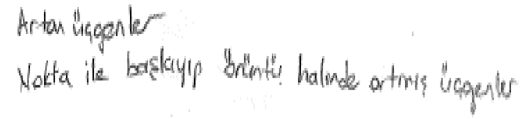
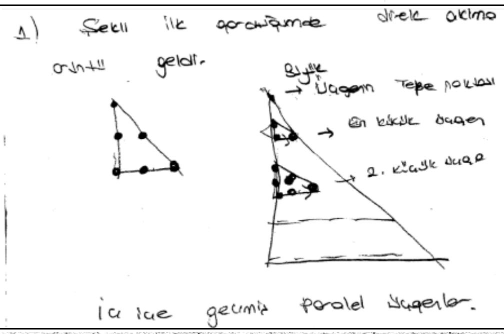
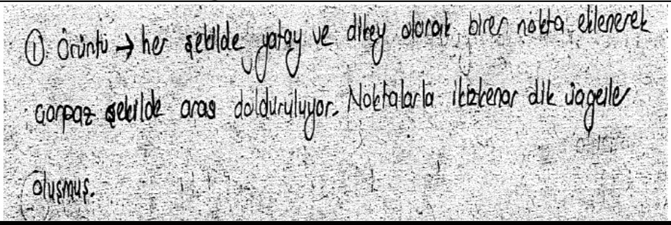
Lastly, three of the preservice teachers stated that a pattern was formed by adding points up to the right or lower side of the previous row. Based on given visual, three of the preservice teachers indicated that a pattern was formed by adding a dot to either the right or lower side of the previous row each time.

Table 4. *Preservice teachers' responses to "Pattern" category*

Category	Rule of Subcategory Pattern	Sample Responses	*f*
Pattern	1, 3, 5,…, 2n-1		1
	Pattern as figure		3
	1, 3, 6, 10,… and amount of increase 1, 2, 3,…		7

Five of the preservice teachers mentioned both the pattern and the formation of triangles. As can be seen from Table 5, the respondent preservice teachers in the "pattern and triangle" category realized that there was a pattern and explained that the rule of this pattern was increasing, isosceles right-angled, or parallel triangles. Therefore, the responses of these preservice teachers were not evaluated in the category of patterns or triangles, but were considered as a separate category.

Table 5. *Preservice teachers' responses to "pattern and triangle" category*

Category	Rule of Subcategory Pattern	Sample responses	*f*
Pattern-triangle	Triangles		2
	Increasing Triangles		1
	Intertwined parallel triangle		1
	Isosceles right-angled triangles		1

The responses of the remaining nine preservice teachers were not found to be similar, and were therefore evaluated in the "other" category. One preservice teacher in the subcategory of "Pascal triangle, right-angled triangle, and Euclidean relation" stated that the visual given were both similar to the Pascal triangle and that triangles can be obtained by combining the dots and that the Euclidean relationship can be taught. Similarly, another preservice teacher likened the visual given to the Pascal triangle. Another preservice teacher expressed generalization. There is also a generalization skill in the pattern. However, since this preservice teacher did not express the pattern, but only emphasized the generalization, hence this response was placed under a separate subcategory. One preservice teacher (see Figure 2) likened the visual to a geometric sequence. However, the expression that the dots that are two times at every turn only three times in the first step indicated that a difficulty in estimating the number of dots given and seeing the rule of the pattern.

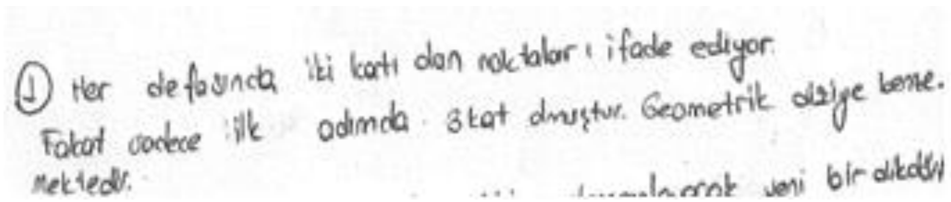


Figure 2. *Preservice teacher's response as "visual is similar to geometric sequence"*

Two of the preservice teachers likened the visual to Binomial expansion, with both establishing a connection with Binomial expansion. Two preservice teachers tried to explain the visual given by the increasing amount, whilst three searched for the relationship between the dots.

Findings from the Second Question

The responses of the preservice teachers to the second visual (see Table 1) are summarized in Table 6. Some of the categories in Table 6 were divided into subcategories.

Table 6. *Preservice teachers' responses to second question*

Category	Subcategory		*f*
No response	No response	7	12
	I can't find – I can't see	2	
	Doesn't mean anything	2	
	I can't comment	1	
Addition	Sum of two odd numbers is even	2	4
	Modeling of 3 + 5	1	
	Modeling of -3 + 5	1	
Area	Area conservation	3	5
	Area calculation by completing a rectangle	2	
Forming figure	Forming new figures by combining figures	7	22
	Forming a rectangle	14	
	Forming a geometric figure by combining the center of circles	1	
Other	Substitution	2	10
	Double figure group	1	
	Form a whole	1	
	Class placement	1	
	Different alignment	1	
	Equation	1	
	Probability – ball pull	2	
	Permutation	1	

In total, 12 of the preservice teachers were unable to write about the given visual. Seven of them left this question blank, one responded as "I could not see," one as "I could not find," and two preservice teachers responded as "It does not mean anything." One preservice teacher mentioned that they could not establish a rhythmical logic as the reason for not making comment. This preservice teacher's response was interpreted as the question not being one that required inductive reasoning skills as in Question 1; that is, the preservice teacher could not see a regular increase and therefore could not make a comment.

From this perspective, it can be said that nonverbal proofs traditionally mentioned in Question 1 are used more widely but that the preservice teachers do not encounter other nonverbal proofs frequently enough. Four of the preservice teachers stated that the given visual was related to the operation of multiplication. Two stated it was modeling that the sum of two odd numbers is an even number, and that the reason for the different colors was that they were brought together in the last figure. The response of one of the preservice teachers is given in Figure 3.

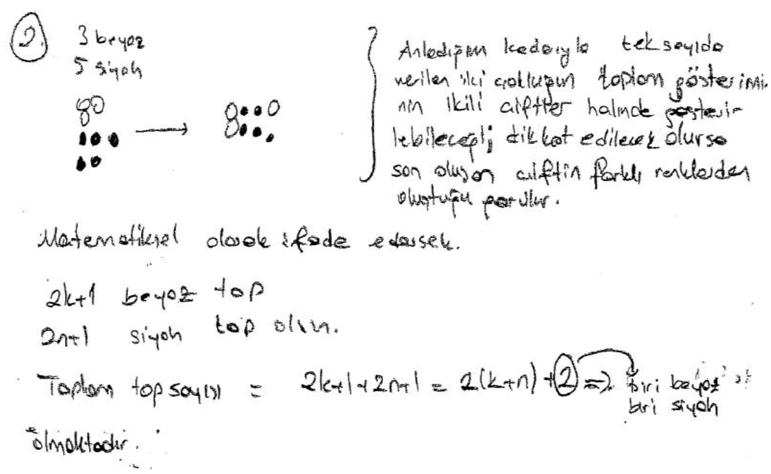


Figure 3. *Preservice teacher's response as "sum of two odd numbers is even"*

As can be seen from Figure 3, this preservice teacher expressed a view based upon the total number of balls being an odd number, as 2k + 1 and the other number as 2n + 1, and thereby found the total number of balls. The other two preservice teachers stated that 3 + 5 and -3 + 5 were modeling. The preservice teacher who stated that it was -3 + 5 modeling considered that the white colored balls were negative and the others were positive, and stated that it was modeling the operation of multiplication by coining.

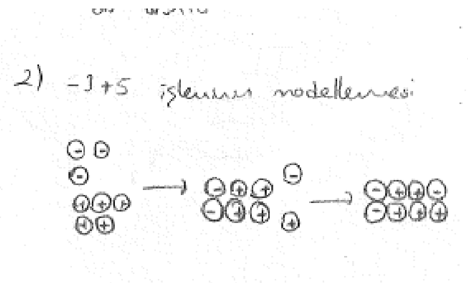


Figure 4. *Preservice teacher's response as "-3 + 5 modeling"*

One of the three preservice teachers stated that *"the area of all geometric shapes is the same even if the parts of the parts change."* This preservice teacher's response was interpreted as he/she thought that the given visual was related to the conservation of the area. Two others stated that the rectangle relates to the field calculation by completing the rectangle.

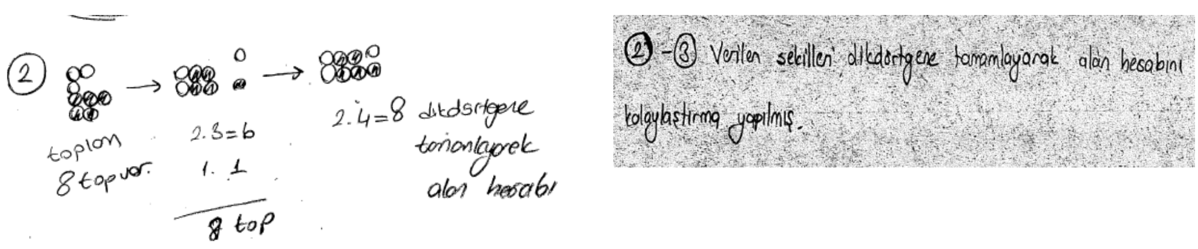


Figure 5. *Preservice teachers' responses as "given visual is related to the field"*

Seven of the preservice teachers thought that the new shape was created by changing the placement of the balls, whist 14 of them stated that it was attempted in order to create a rectangle. Here, the preservice teachers described the shapes given in the visuals as jigsaws, objects, filled/empty circles, filled/empty circulars, rounds, rings, and black/white balls. This was interpreted as the existing schemas of the given visualization were very different. One preservice teacher stated that unlike the others, it was related to creating a triangle, square, pentagonal, trapezoid, and parallelogram by merging the centers of the circles.

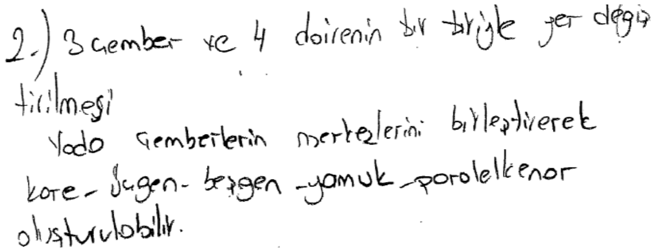


Figure 6. *Preservice teacher's response as "related to creating a geometric shape by merging the center of circles"*

The response of the remaining 10 preservice teachers was evaluated under the "other" category. Two preservice teachers mentioned that only the balls were replaced. One preservice teacher stated that it was attempted to create only one or two groups, another stated that it was to connect pieces to see the whole, one stated that it represented the females as the black balls and the males as the white balls in the classroom, so one female, and one male had to sit in the classroom, whilst one preservice teacher mentioned the different array of the balls. One preservice teacher stated seeing the equations with a shape.

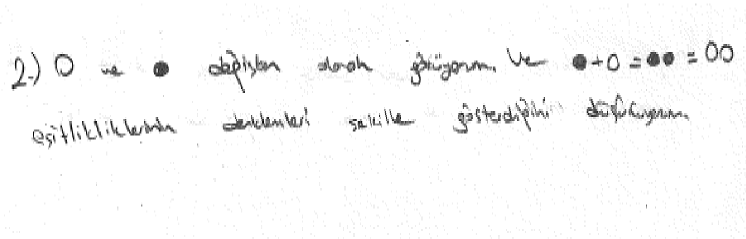


Figure 7. *Preservice teacher's response as "related to the equation"*

As can be seen from Figure 7, the preservice teacher considered the black and white shapes to be variables such as x and y, and as a result wrote x + y = x + x = y + y. However, this is only possible if x = y. Three preservice teachers stated thinking like a black or white ball and that it could be related to permutation, or ball drawing probability.

Findings from the Third Question

The responses of the preservice teachers to the third image (see Table 1) directed to the preservice teachers are summarized in Table 7. Some of the categories in Table 7 were divided into subcategories.

Table 7. *Preservice teachers' responses to third question*

Category	Subcategory	*f*	
No response		1	1
Modeling sample	Modeling of operation of multiplication	3	4
	Modeling of exponential expression of numbers	1	
Creating a figure	Creating a rectangle	21	23
	All geometric shapes can be obtained	2	
Other	Area	12	25
	Abacus	2	
	Problem question	1	
	Convert to symmetric format	1	
	Number of balls	5	
	Smoothing the shape	1	
	Replacement of balls	3	

As can be seen in Table 7, only one of the preservice teachers did not respond. Four of them stated that the visual might be related to modeling, similar to the second question. The majority of the preservice teachers perceived the visual as creating new shapes by way of the replacement of points. When the responses were examined, it was perceived as creating a rectangle by transforming from the rectangle above, trapezoid, and especially a right trapezoid. Along with this, they also stated that it could be used to calculate the area.

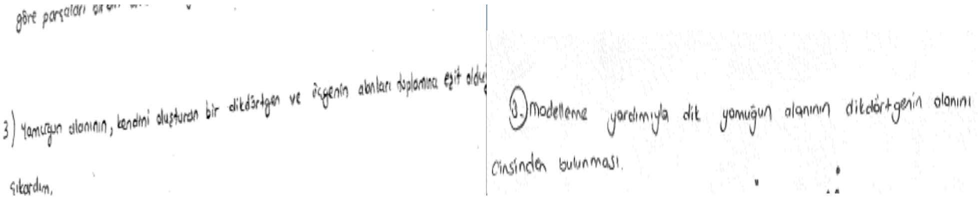


Figure 8. *Preservice teachers' response as "Creating a figure"*

The responses of the other preservice teachers were based on replacement of the points.

DISCUSSION

There are many studies on the role of visualization and visual thinking in mathematics teaching. Thornton (2001) stated that visualization provides mathematical rules, simple, elegant, powerful approaches to extracting results, and may even be an effective method in the classroom for students with different learning styles. For this reason, the ability of preservice teachers to work with visual proofs and visual proving is becoming an important dimension. The purpose of the current study was to examine these skills. Only 12 of the preservice teachers who participated in the study stated that the image given in Figure 9 related to "the sum of numbers from 1 to n."

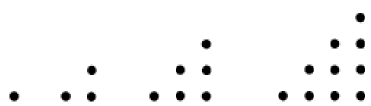
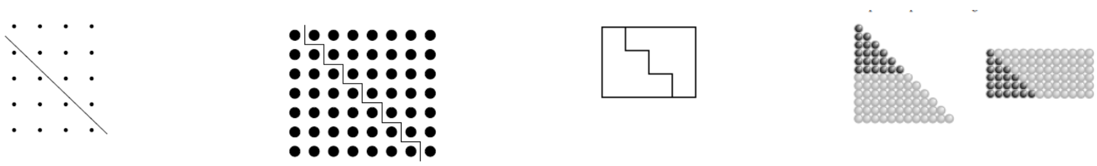


Figure 9. *Image of first question used in the study*

Four of these preservice teachers explained the visual as "the sum of numbers from 1 to n," while eight of them tried to verify the number of points using the formula $\frac{n\,(n+1)}{2}$ for each unit step. However, none of the preservice teachers made a connection between the given visual and the formula $\frac{n\,(n+1)}{2}$. This example can be seen in many sources and undergraduate courses. When the responses in this category were examined, it was seen that the preservice teachers were unable to express using the visual $\frac{n\,(n+1)}{2}$, even though they indicated that the given visual relates to the sum of numbers from 1 to n. It was interpreted that they had previously encountered this visual, but that they were unfamiliar with activities related to nonverbal proving skills.

A total of 14 of the preservice teachers likened the visual to a triangle (isosceles and right-angled triangles etc.) based on the given shape. 15 of the preservice teachers realized that it formed a pattern. The "other" category included the responses of 11 preservice teachers. Although it is a frequently encountered visual, the preservice teachers were unable to establish a relationship between the given visual and the sum of numbers from 1 to n. The preservice teachers who established a connection tried to verify the rule on the visual instead of using the visual to explore the formula. Of course, it is possible to use different visuals to help prove the same expression. Pease, Colton, Ramezani, Smaill, and Guhe (2010) expressed that in using the triangular numbers for the sum of the consecutive numbers, the visual in Figure 10a could be obtained. Similarly, Figure 10b was given by Miller (2012), in which n and (n + 1) can be in the form of half the area of a rectangle with side length. Additionally, Figure 10c is sourced from the study of Britz, Mammoliti, and Sørensen (2014) and Figure 10d from Alsina and Nelsen (2010).



a) b) c) d)

Figure 10. *Visuals related to the sum of consecutive numbers*

Of course, the visuals given in Figure 10 were all directed to the same expression. However, each shape could be difficult to perceive when given in this form if the student or preservice teacher had not previously encountered them. Miyazaki (2000) further explained these as shown in Figure 11.

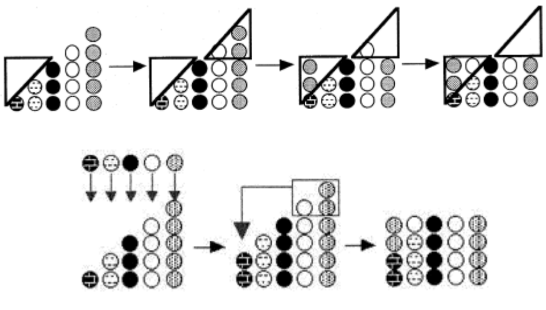


Figure 11. *Visual proof given by Miyazaki (2000)*

As can be seen in Figure 11, explaining what the nonverbal proofs are and how they are thought can make it easier to understand.

The third question of the study (see Figure 12) relates to a similar image, but rather than starting from 1 it starts from 2. The idea being to see whether or not the similarity would be established with the visual from the first question, related to the proof of another expression, or would it be customized or generalized. In short, this question was asked in order to compare the thinking processes of the participant preservice teachers.

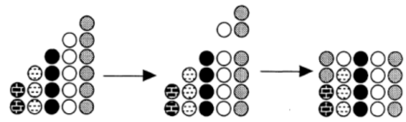
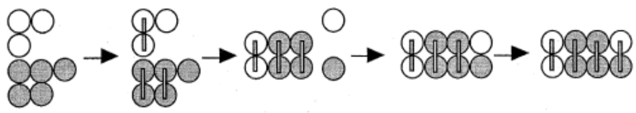


Figure 12. *Image of third question used in the study*

When given in this form, none of the preservice teachers were able to establish a relationship between the sum of consecutive numbers, or the sum of five consecutive numbers in the special case – being five times the number in the middle that was $2+3+4+5+6=4.5$. In fact, it is a rather simple proof when given the rule and asked to prove; however, it becomes difficult to understand when simply working from the visual proof.

Although the first and third questions were similar, none of the preservice teachers managed to combine or link the two visuals. The difference between these two visuals as a thinking processes can be interpreted as causing different perceptions and are therefore non-interpretable; a view supported by Pease et al. (2010). In fact, Pease et al. (2010) expressed that the visual given in Figure 10a was illustrating the visual status of $1+2+3+4=4*5/2$, and based upon this particular case generalization was applied to the expression "the sum of the first n consecutive number is $n.(n+1)/2$." Based on such a visual, proving not only requires mathematical thinking, but can be a difficult skill for students, preservice teachers or even serving teachers who have not encountered visualization. Another difficulty here is the generalization problem, as Kulpa (2009) stated. It is a problem of not perceiving an idea about a rule from a few special cases.

The second question directed to the preservice teachers was similar to the first question, since it was considered that preservice teachers would remember the modeling examples, as well as the third question since it is related to customizing skill. In a different expression, while inductive thinking plays an important role in the first question, the second question was expected to give the visual and generalize it for a special case as in many proofs according to Nelsen. As Alsina and Nelsen (2010) stated, the aim in this nonverbal proof is to provide an understanding by visualizing mathematical expression for any given situation. Customization, generalization, proof, and assumption are sub-dimensions of mathematical thinking skills. Therefore, the findings obtained from these questions also provide important evidence about mathematical thinking processes. The findings obtained from this question showed that only four of the preservice teachers stated that it could be modeling examples related to the given visual, whilst others made no comments, either associating them with the field or attempting to create a shape. One important finding is that in both questions, the preservice teachers generally perceived each step as creating a new shape. At the same time, it was observed that it was associated with the area that does not change with the replacement. The second question has been described by Miyazaki (2000) as follows (see Figure 13). Here, both the formal and nonverbal proof is given one under the other.



$$X+Y \rightarrow (2n+1)+(2m+1) = 2n+2m+1+1 = 2n+2m+2 = 2(n+m+1)$$

Figure 13. *Visual proof given by Miyazaki (2000)*

It would be perhaps easier to make a comment when given in this form. Indeed, as Alsina and Nelsen (2010) stated, "*Proofs without words are pictures or diagrams that help the reader see why a particular mathematical statement may be true, and also see how one might begin to go about proving it true*" (p. 118). In the study of Doruk and Güler (2014), it was determined that participant self-confidence was low in proving and in the understanding of proofs. The inclusion of nonverbal proofs can help students increase their self-confidence in proving. On the other hand, as stated by Altıparmak and Öziş (2005), teaching and development of students' proof and reasoning skills are dependent on the teacher. If teachers afford their students a wide range of learning opportunities and provide different proof methods, the students will likely better understand mathematics and logical thinking, and thereby increase their level of creativity. Based on this thought, nonverbal proofs can be an effective tool. However, as the teachers of the future, preservice teachers are recommended to have a level of experience with nonverbal proofs.

Another important factor is a lack of visual reasoning skills. Iannone and Inglis (2011) showed the same question in a university setting in the manner of "Prove a mathematical argument"; and the results showed that it was difficult for students to produce a correct argument rather than recognizing the proof given in the deductive argument. It can be said that nonverbal proofs may help in this sense to develop proofing skills or even to produce new arguments or proving. Bardelle (2009) explained the reasons behind a lack of visual reasoning skills as, "*poor knowledge of certain basic mathematical tools, poor acquaintance with the use of figural representations, a conflict between the conceptual and perceptual nature of diagrammatic proofs and sometimes poor understanding of the concept of mathematical proof itself*" (p. 259). Reasoning and proof are one of the process skills areas that students, as well as prospective and serving teachers need to master. As Bell (2011) stated, it is important to provide students with opportunities to improve their process skills and to help them understand the processes of proof. At the same time, Bell stated that using nonverbal proofs could help students to understand the process of proof. Moreover, Bell stated that they could improve reasoning skills so that students were better able to understand how to start a formal proof. In this context, the importance of proof in mathematics teaching at every level should be emphasized and it is recommended that students, especially in undergraduate education, gain experience with nonverbal proofs.

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TÜRKÇE GENİŞ ÖZET

Matematik Öğretmen Adaylarının Sözsüz İspat Becerilerinin İncelenmesi: Bir Durum Çalışması

Handan DEMİRCİOĞLU[2]

GİRİŞ

Matematiği öğrenmede ve öğretmede hem ispatın hem de görselleştirmenin öneminin vurgulanmasına paralel olarak sözsüz ispatlar son yapılan çalışmalarda hem ispat yapmayı hem de ispat becerisini destekleyen yöntem olarak karşımıza çıkmaktadır. İlköğretimden üniversite matematiğine kadar her düzeyde karşımıza çıkan sözsüz ispatlar belli bir matematiksel ifadenin *niçin* doğru olabildiğini görmesine ve ayrıca o ifadenin ispatlamaya *nasıl* başlayabileceğini görmeye yardım eden resimler veya diyagramlar olarak tanımlanmaktadır (Alsina ve Nelsen, 2010). Bir resim bin kelimeden daha iyidir (Thornton, 2001; Rösken ve Rolka, 2006). Casselman'ın (2000) ifade ettiği gibi bu söz birçok kültürde ortaktır. Aslında diyagramlarla-resimlerle ispatlar uzun bir tarihe sahiptir (Foo, Pagnucco ve Nayak, 1999). Tarihsel süreçte çok kullanılmasına rağmen, matematik öğretiminde kullanımı oldukça yenidir.

Gerçekten Stucky (2015) sözsüz ispatların faydalı bir pedagojik araç olduğunu ifade etmiştir. Sözsüz ispatlar öğrenciler için klasik ispatlardan daha çok ilgi çekici ve kabul edilebilirdir (Štrausová ve Hašek, 2013), öğrencilerin ispat sürecini anlamada etkilidir (Bell, 2011) çeşitli matematiksel özellikleri anlama sürecinde önemli rol oynarlar (Štrausová ve Hašek, 2013).

Literatür incelendiğinde yapılan çalışmalar genellikle sözsüz ispatların kuramsal yapısı, tarihçesi ve örneklerine (Alsina ve Nelsen, 2010; Miller, 2012) sınıf içi uygulama örneklerine (Bell, 2011; Gierdien, 2007), sözsüz ispat uygulamalarında öğrenci zorluklarına (Bardelle, 2009), yetenekli ortaöğretim öğrencilerinin sözsüz ispat oluşturma yaklaşımlarına (Uğurel, Moralı ve Karahan, 2011), sınıf öğretmeni adaylarının somut modeller ile ispat yapma ile ilgili görüşlerine (Doruk, Kıymaz ve Horzum, 2012) yöneliktir. Demircioğlu ve Polat (2015; 2016) öğretmen adaylarıyla yapmış oldukları çalışmada öğretmen adayları sözsüz ispatların ispat, problem çözme, anlama, zihinsel, akıl yürütme, genelleme, işlem, analiz ve sentez yapabilme, görme ve düşünme becerilerini kazanmada etkili olduklarını ifade etmişlerdir. Aynı zamanda sözsüz ispatlarla ile en fazla zorlandıkları yerlerin verilen şekilleri anlayamama, açıklama olmaması, sözsüz ispat ile cebirsel ispat arasında ilişki kuramama, alan bilgisi eksikliği, kaynak sıkıntısı olduğunu göstermiştir. Gerçekten sözsüz ispatlarda yalnızca bir görsel kullanıldığı için, gördüğü zaman tanıyabilme, görseli kullanarak ispat yapabilme, yorum yapabilme önemli bir beceri olarak karşımıza çıkmaktadır. Bu nedenle bu çalışmada hem matematik öğretmen adaylarının sözsüz becerilerini incelemek ve elde edilen bulgular doğrultunda öneriler sunmak amaçlanmıştır.

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YÖNTEM

Çalışma nitel araştırma desenlerinden durum çalışması ile yürütülmüştür. Çalışmanın katılımcılarını İç Anadolu'da bulunan bir devlet Üniversitesi'nde öğrenimlerine devam eden son sınıftaki 53 öğretmen adayı oluşturmaktadır. Veriler öğretmen adaylarına yöneltilmiş olan 3 sözsüz ispat örneği ile yazılı olarak toplanmıştır. Bu sorulardan birincisi birçok kaynakta yer verilen 1'den n'ye kadar sayıların toplamının modellemesine örnek olarak verilen örnektir.2. ve 3. soru ise Miyazaki'den (2000) alınmıştır. 2. soru "iki tek sayının toplamı çifttir" ve 3. soru ise ilk soruya benzemekle birlikte farklı ifadelerin ispatı içinde kullanılabilmektedir. Verilerin analizi öğretmen adaylarının cevaplarının benzerlik ve farklılıklarına göre sınıflandırılarak yapılmıştır. Kategori ve alt kategorilerin elde edilme sürecinde öncelikle verilen cevaplarda yer verilen ifadeler direk alınarak kodlanmıştır. Yani "1'den n kadar sayıların toplamı", "Üçgen", ,… vb gibi kodlanmış ve frekans tablosu oluşturulmuştur. Daha sonraki aşamada ise her bir kategori kendi içinde ele alınarak alt kategoriler oluşturulmuştur. Bu şekilde elde edilen bu kategori ve alt kategoriler alanında uzman iki farklı araştırmacı tarafından ayrı ayrı ve sonra birlikte incelenmiştir. Bu aşamadan sonra kategorilere son hali verilmiştir. Katılımcıların yazılı ifadelerinden direk alıntılar yapılarak, verilerinin güvenilirliği artırılmıştır.

BULGULAR

Elde edilen bulgular öğretmen adaylarının verilen görselleri genellikle geometrik şekillerle ilişkilendirdiklerini göstermiştir. Ayrıca verilen görsel ile matematiksel ifade arasındaki ilişkiyi görenlerin ise görseli ispat yapmak için kullanmak yerine ifadenin doğru olduğunu göstermek için kullandıkları görülmüştür. İlk yöneltilen 1'den n'ye kadar olan sayıların toplamı ile ilgili olan görsel, birçok kaynakta ve birçok lisans dersinde sıklıkla verilmektedir. Fakat sıklıkla karşılaşılan bir görsel olmasına rağmen öğretmen adayları verilen görsel ile 1'den n'ye kadar olan sayılar arasında ilişki kuramamıştır. İlişki kuran öğretmen adayları ise formülü keşfetmek için şekli kullanmak yerine kuralı görsel üzerinde doğrulamaya çalışmışlardır. İkinci sorudaki görsel ilk görsele benzerdir fakat 1'den başlatmayıp 2'den başlatılmıştır. İlk soru ile benzerlik mi kurulacak yoksa başka bir ifadenin ispatı ile mi ilişkilendirilecek ya da özelleştirme veya genelleştirme mi yapılacak görülmek istenmiştir. Kısaca öğretmen adaylarının düşünme süreçleri karşılaştırılmak istenmiştir. Birinci ve üçüncü sorular benzerlik göstermesine rağmen iki görseli birleştiren hiçbir öğretmen adayı da olmamıştır. Bu iki görseldeki düşünme süreçleri olarak farklılık göstermesi farklı algılanmasına, yorum yapılamamasına neden olmuştur şeklinde yorumlanabilir. İlerideki çalışmalarda aynı ifadenin ispatı için farklı görselleri aynı ölçme aracında vererek düşünme süreçleri incelenebilir. Ayrıca $1+2+3+4=4*5/2$ özel durumunu gösteren görselden yola çıkılarak "ilk n ardışık sayının toplamının $n(n+1)/2$" ifadesine genelleştirme yapıldığı görselleştirmeler ilk defa karşılaşan öğrenciler, öğretmen adayları hatta öğretmenler için bile zor bir beceri haline gelmektedir. Burada diğer bir güçlük de genelleştirme problemidir. Özel birkaç durumdan hareketle kural hakkında bir fikir elde edememe problemidir. Bu nedenle kuralların öğretiminde veya kuralların ispatları yapılırken bu düşünme veya ispat yöntemini de bir pedagojik araç olarak kullanmak hem matematiksel düşünmenin gelişimine hem ispat becerisinin gelişimine hem matematiği anlamlı öğrenmeye yardımcı olacaktır.

Anahtar Sözcükler: İspat, Sözsüz ispat, Görsel ispat