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Elliptic quaternion and elliptic linear interpolation

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Elliptic Quaternion and Elliptic linear Interpolation

Highlights

- *linear elliptic interpolation (Elerp)*
- *spline elliptic quaternion interpolation (Esquad)*
- *cubic interpolation out of three linear interpolations*
- *consistently and continually differentiable Esquad*
- *interpolation between a series of position and direction interpolations on the ellipsoid.*

Graphical Abstract

Elliptic quaternion interpolation between the four key frames on ellepsoid.

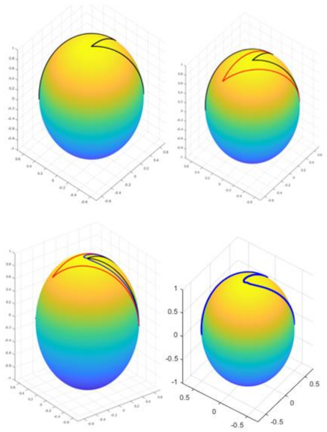

Figure. An illustration of the study

Aim

Aim of this work is to seamlessly interpolate between a series of position and direction interpolations on the ellipsoid.

Design & Methodology

Idea is to make a choice between s_n and s_{n+1} elliptic quaternions in order to allow control of endpoint derivatives in spline segments. Thus, we showed that ESquad is consistently and continuously differentiable across all segments.

Originality

In this paper, we have been done elliptic quaternion linear interpolation on ellipsoid using elliptic quaternions.

Findings

We showed that ESquad is consistently and continuously differentiable across all segments.We seamlessly interpolated between a series of position and direction interpolations on the ellipsoid.

Conclusion

We presented the spline elliptic quaternion interpolation on the ellipsoid in this paper.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Elliptic Quaternion and Elliptic linear Interpolation

Araştırma Makalesi / Research Article

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ABSTRACT

Spherical spline quaternion interpolation has been done on sphere in Euclidean space using quaternions. In this paper, we have been done elliptic quaternion linear interpolation on ellipsoid using elliptic quaternions. This interpolation curve is called Elerp elliptic linear interpolation. In addition, ESquad (spline elliptic quaternion interpolation) is defined by using the group structure feature of elliptic quaternion on ellipsoid.

Keywords: Quaternions, elliptic quaternions, interpolation, euclidean space, elerp, spline, esquad.

1. INTRODUCTION

Quaternions were described by William Rowan Hamilton in 1843. Quaternions with non-commutative real algebra structure have a similar structure to complex numbers since they carry most of the properties of complex numbers. Nowadays, quaternions are used especially in optimization problems involving physics, kinematics, computer graphics, animation and rigid body transformations. Shoemak [10] defined the geodesic curve between two points on the sphere with the help of quaternions, and this is called the spherical linear interpolation Slerp on the sphere. In terms of differential geometry, great arcs are segments of geodesic curves on the sphere. Slerp not only creates great arcs on the unit sphere, but also creates the shortest arcs. Thus, Slerp gives the interpolation path using the shortest arc between two quaternions on the unit sphere. Robot kinematics and computer games are covered extensively in Slerp notation. Especially in recent years, spherical interpolation and interpolations in Minkowski space and ellipsoid space have an important place in animation and modeling of robot movements in 3-dimensional computer games [12]. Also, interpolation between two rotations (slerp) is optimal. But when interpolating between a number of rotations, the following issue appears:

At the control points, the curve is not smooth. Smooth interpolation is used in computer animation to model motion solids, cameras, and lights [7,12]. The work done so far using quaternions has been interpolated on the Euclidean sphere, Lorentz and Hyperbolic spheres [4, 11]. In this study, interpolations is made on the ellipsoid using the elliptic quaternions. Also, the aim of this work is to seamlessly interpolate between a series of position and direction interpolations on the ellipsoid.

2. MATERIAL AND METHOD

In the proposed method, the curve is not smooth at the control points in a series of rotations. This smoothness issue is not easy to solve. Similarly, interpolation is easy with a straight line in the plane between two points. But even in simple Euclidean space it is complicated to properly interpolate a set of points. Typically, in the plane of cubic curves different types of interpolations between a set of control points are used. For a number of elliptic quaternion $\{q_n\}_{n=0}^{N-1}$, the algorithmic expression for Esquad produces an interpolation curve. Under the conditions that the spline pass through the control points and that the derivatives are continuous, we introduced a spline structure that interpolated those elliptic quaternions. Idea is to make a choice between s_n and s_{n+1} elliptic quaternions in order to allow control of endpoint derivatives in spline segments. Thus, we showed that ESquad is consistently and continuously differentiable across all segments. In other words, we seamlessly interpolated between a series of position and direction interpolations on the ellipsoid.

3. ELLIPTIC QUATERNIONS

With the help of a little generalization in real quaternions, it is possible to examine the rotation and interpolation on the elliptic instead of the sphere. For this purpose, the dot product and the vector product must be defined in accordance with the metric that accepts the elliptic sphere. For each elliptic, a corresponding quaternion can be defined. We will call this quaternion the elliptic quaternion. Elliptic quaternions are a 2-dimensional vector space on the set of elliptic numbers and a 4-dimensional vector space on the set of real numbers [13].

In the R^3 space, $a_1, a_2, a_3 \in R^+$

$E: x^2 + a_2 y^2 + a_3 z^2 = 1$

Consider the ellipsoid and denote it with $\sqrt{a_1 a_2 a_3} = \Delta$

$$H_{a_1,a_2,a_3} = \{q_1 + q_2 i + q_3 j + q_4 k : q_i \in R, i^2 = -a_1, j^2 = -a_2, k^2 = -a_3, ijk = -\Delta\} \quad (1)$$

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The set that is not commutative, but associative, defined in its form, is called the set of elliptic quaternions corresponding to the ellipsoid " E".

4. ELLIPTIC DOT PRODUCT AND VECTOR PRODUCT OF AN ELLIPSOID

The dot product of an $E: a_1x^2 + a_2y^2 + a_3z^2 = 1$ ellipsoid $v = v_1i + v_2j + v_3k$ and $w = w_1i + w_2j + w_3k$ in the form of

$$\beta(v, w) = a_1v_1w_1 + a_2v_2w_2 + a_3v_3w_3 \tag{2}$$

the elliptic vector product is also defined as [13]

$$\vartheta(v \times w) = \Delta \begin{vmatrix} -i/a_1 & j/a_2 & k/a_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{3}$$

Definition 1. For the elliptic set of quaternions H_{a_1,a_2,a_3} and $p, q \in H_{a_1,a_2,a_3}$ Elliptic quaternion product of $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ elliptic quaternions [13] is defined as,

$$pq = p_1q_1 - \beta(v_p, v_q) + p_1v_q + q_1v_p + \vartheta(v_p \times v_q) \tag{4}$$

Definition 2. For the elliptic set of quaternions H_{a_1,a_2,a_3} and $q \in H_{a_1,a_2,a_3}$ and $q = S_q + V_q$, $S_q = 0$, then q is called a purely elliptic quaternion. The product of two purely elliptic quaternions [13] is expressed as,

$$pq = -\beta(v_p, v_q) + \vartheta(v_p \times v_q)$$
$$= -(a_1p_2q_2 + a_2p_3q_3 + a_3p_4q_4) + \Delta \begin{vmatrix} i/a_1 & j/a_2 & k/a_3 \\ p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \end{vmatrix} \tag{5}$$

Definition 3. $q \in H_{a_1,a_2,a_3}$ $q = (q_1, q_2, q_3, q_4) = S_q + V_q$ being an elliptic quaternion, quaternion conjugate and norm and inverse, are respectively defined as,

$$\bar{q} = S_q - V_q \tag{6}$$

$$\bar{q} = q_1 - q_2 - q_3 - q_4 \tag{7}$$

$$N_q = \|q\| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_1^2 + a_1q_2^2 + a_2q_3^2 + a_3q_4^2} \tag{8}$$

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}, N_q \neq 0 \tag{9}$$

In addition, if $N_q = 1$, then q units are called elliptic quaternions [13][14].

Definition 4. Elleptic quaternions can be expressed in polar form as well as complex numbers and quaternions.

$$\cos\varphi = \frac{q_1}{\|q\|} \ \ ve \ \ \sin\varphi = \frac{\sqrt{a_1q_2^2 + a_2q_3^2 + a_3q_4^2}}{\|q\|} \tag{10}$$

can be written in the form of,

$$q = \|q\|(\cos\varphi + m\sin\varphi) \tag{11}$$

Here

$$m = \frac{(q_2, q_3, q_4)}{\sqrt{a_1q_2^2 + a_2q_3^2 + a_3q_4^2}} \tag{12}$$

Vector m is a unit vector in the space R^3 with dot product β. In addition, this vector is $m^2 = -1$ according to the product of the elliptic quaternion [13].

Theorem 1. Each $q \in H_{a_1,a_2,a_3}$ $q = q_1 + q_2i + q_3j, q_4k = (\cos\varphi + m\sin\varphi)$ Elliptic unit quaternion corresponds to a rotational motion on the ellipsoid

$$a_1x^2 + a_2y^2 + a_3z^2 = 1 \ . \tag{13}$$

The dot product of this ellipsoid is β_{a_1,a_2,a_3}.

For a point b on this ellipsoid, the linear transformation

$$R_q(b) = qbq^{-1} \tag{14}$$

indicates an elliptic rotation by angle 2φ on the plane perpendicular to the m-axis with respect to the dot product of β_{a_1,a_2,a_3} [13].

5. RESULTS AND DISCUSSION

Elerp (Elliptic Linear Interpolation)

In this section, interpolation is calculated on the ellipsoid. This interpolation is done using elliptic quaternions. These interpolation curves are defined as linear elliptic interpolation (Elerp) and spline elliptic quaternion interpolation (Esquad).

Definition 5. Let's take $q = [\cos\varphi, m\sin\varphi] \in H_{a_1,a_2,a_3}$ and $m \in R^3$. In this case, the logarithm m function is defined as,

$$\log q \equiv [0, \varphi m] \tag{15}$$

Lema 1. Let's take $= [\cos\varphi, m\sin\varphi] \in H_{a_1a_2a_3}$ and $n \in R$. In this case

$$\frac{d}{dn}q^n = q^n \log(q) \tag{16}$$

Lema 2. Let's take $q \in C^1(R, H_{a_1a_2a_3})$, $r \in C^1(R, R)$. In this case $q(t) = [\cos\ \varphi(t),\ m(t)\sin\varphi(t)]$. So $\frac{d}{dt}q(t)^{r(t)} =$

$$\begin{bmatrix} \sin(r(t)\varphi(t))\left(r'(t)\varphi(t) + r(t)\varphi'(t)\right), & \cos(r(t)\varphi(t)) \\ \left(r'(t)\varphi(t) + r(t)\varphi'(t)\right) m(t) + \sin(r(t)\varphi(t))\ m'(t) & \end{bmatrix} \tag{17}$$

Description 6. The product of $p^{-1}q$ eleptic quaternions can be greatly simplified by using the elliptic unit quaternions $w = [\cos\varphi, m\sin\varphi]$ and $w^t = [\cos t\varphi, m\sin t\varphi]$.

Definition 7. The elliptic quaternion used for a rotation denoted by p and ending in q is $p, q \in H_{a_1a_2a_3}$, $q = p(p^{-1}q)^n$. So $Elerp\ (p, q, n)$ is expressed

$$Elerp\ (p, q, n) = p(p^{-1}q)^n, \ \ n \in [0,1] \tag{18}$$

Lema 3. Let $p, q \in H_{a_1a_2a_3}$. In this case, $Elerp(p, q, n), n \in [0,1]$ forms a great arc with the shortest length on the ellipsoid between p and q unit elliptic quaternions.

Lema 4. When p and q are two-unit elliptical quaternions on various elliptic surfaces, $p \in H_{a_1a_2a_3}$ and $q \in H_{b_1b_2b_3}$; then the product of elliptic quaternions cannot describe the structure of the associated rotation. Each elliptic quaternion has a unique scalar product space, which causes variations in the elliptic quaternion product.

The unit elliptic quaternions create the shortest great arc on the ellipse, according to the interpolation curve for elliptic linear interpolation. Elerp, which also has a scalar parameter n that indicates how far to interpolate between two elliptic quaternion turns, p and q, is typically thought to be the best interpolation curve between them. (Where n is a real number, n=0 gives us p, n=1 gives us q, and intermediate values of n give us the elliptic quaternions on the path between p and q). It should be noted that Elerp only permits interpolation between two rotations. It is required to take into account several, difficult interpolations when taking into account more than two rotations.

Definition 8. Definition of Elerp for the elliptic quaternion $p, q \in H_{a_1,a_2,a_3}$,, used for a rotation denoted by the initial p and ending with q can be given as,

$$Elerp(p,q,n) = \frac{p\,sin((1-n)\varphi)+q\,sin(n\varphi)}{sin(\varphi)}$$

$$p, q \in H_{a_1,a_2,a_3}, \quad n \in [0,1] \tag{19}$$

Here, $p \cdot q = \cos\varphi$.

Remark 2. Elerp for unit eleptic quaternions is written as,

$$Elerp(p,q,n) = p(p^{-1}q)^n \tag{20}$$

Proof:

$$q = p(p^{-1}q)^n \tag{21}$$

Which is $p^{-1}, q \in H_{a_1,a_2,a_3}$,, then $p^{-1}q \in H_{a_1,a_2,a_3}$. In this case

$$p^{-1}q = cos\,\varphi + m\,sin\,\varphi \tag{22}$$

The angle between p and q is shown here as φ. So that the adjustment of p changes evenly along the great arc between p and q, parameter n can be introduced into the angle.

In that case,

$$q(n) = p[cos(n\varphi) + m\,sin(n\varphi)] = p[cos\,\varphi + m\,sin\,\varphi]^n = p(p^{-1}q)^n \tag{23}$$

$$\frac{p\,sin((1-n)\varphi)}{sin\,\varphi} = \frac{p(sin\,\varphi\,cos(n\varphi) - cos\,\varphi\,sin(n\varphi))}{sin\,\varphi} = p\,cos(n\varphi) - \frac{p\,cos\,\varphi\,sin(n\varphi)}{sin\,\varphi} \tag{24}$$

According to equation (22)

$$\frac{q\,sin(n\varphi)}{sin\,\varphi} = \frac{p(cos\,\varphi + m\,sin\,\varphi)\,sin(n\varphi)}{sin\,\varphi} = \frac{p\,cos\,\varphi\,sin(n\varphi)}{sin\,\varphi} + \frac{pm\,sin\,\varphi\,sin(n\varphi)}{sin\,\varphi} \tag{25}$$

Adding the equations' sides (24) and (25)

$$\frac{p\,sin((1-n)\varphi) + q\,sin(n\varphi)}{sin(\varphi)} = p\,cos(n\varphi) - \frac{p\,cos\,\varphi\,sin(n\varphi)}{sin\,\varphi} + \frac{p\,cos\,\varphi\,sin(n\varphi)}{sin\,\varphi} + \frac{pm\,sin\,\varphi\,sin(n\varphi)}{sin\,\varphi}$$

In this case

$$\frac{p\,sinh((1-n)\phi) + q\,sinh(n\phi)}{sinh(\phi)} = p[cos\,\varphi + m\,sin\,\varphi]^n = p(p^{-1}q)^n \tag{26}$$

Equation (23) is a simple application of Elerp's derivative (16),

$$Elerp'(p,q,n) = p(p^{-1}q)^n\,log(p^{-1}q). \tag{27}$$

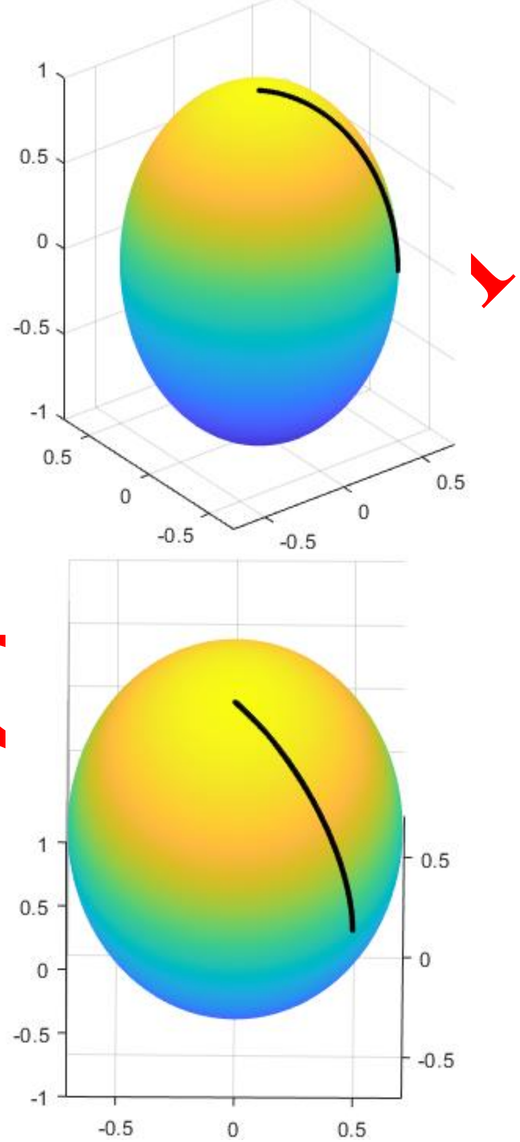


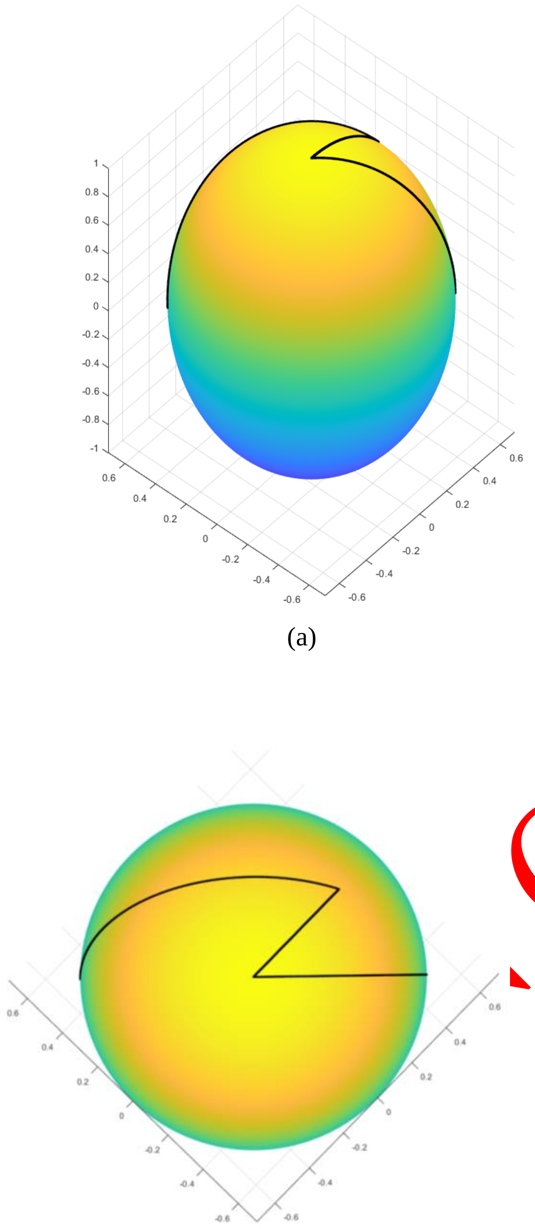
Figure 1. MATLAB Programming Language simulates the interpolation shapes.

50 interpolated frames make up the interpolation curve between two elleptic quaternions on the ellepsoid

Over a Series of Rotations, Interpolation

The best interpolation is between two rotations (Elerp). But when a series of rotations are interpolated, the curve is not smooth at the control points.

Continuity throughout the entire interpolation can easily be achieved with a reparametrization. Since the interpolation parameter is actually transformed into a sequence of discrete frames in between each pair of key frames, reparametrization actually refers to figuring out how many frames should be included in each interval based on the interval's size. The size of an interval can be measured by the angle $\cos\varphi = q_i \cdot q_{i+1}$ between two pairs of key frames q_i and q_{i+1}. (see figure 2).



(a)

(b)

Figure 2. There are 50 interpolated frames in an elliptic quaternion interpolation between four key frames;

This smoothness issue is not easy to solve. Similarly, the interpolation it is easy with a straight line in the plane between the points. However, it is challenging to correctly interpolate a set of points even in simple Euclidean space. Different types are typically used in interpolations between a set of control points in the plane of cubic curves. In quaternion space three linear interpolations should be used to create a cubic interpolation. A quantity determined by the logistic equation $2n(1-n)$ is interpolated between the initial data point and two additional (well chosen) points, followed by the remaining points. If auxiliary points are selected properly their continuity can be achieved. A cubic elliptic interpolation (unit elieptic quaternion) between data points of the esquad function is determined by the points q_1 and q_2 and the quantity $n \in [0,1]$ as follows.

$$Esquad(q_i, q_{i+1}, s_i, s_{i+1}, n) = Elerp(Elerp(q_i, q_{i+1}, n), Elerp(s_i, s_{i+1}, n), 2n(1-n)) \quad (28)$$

The s_i and s_{i+1} inner quadrangle points are called inner quadrangle points and thus these points must be chosen carefully to ensure the continuity in the segments.

Spline Interpolation of Elliptic Quaternion

Considering the algorithmic expression for the esquad is a set of $\{q_n\}_{n=0}^{N-1}$ eleptic unit quaternions, we want to construct a spline curve by interpolating the eleptic quaternion with conditions whose derivatives are continuous and pass through control points. Idea is to make a choice between s_i and s_{i+1} elliptic quaternions to allow control of endpoint derivatives in spline segments. Easily with squad definition

$$R_i(n) = Esquad(q_i, q_{i+1}, s_i, s_{i+1}, n) = Elerp(Elerp\ (q_i, q_{i+1}, n), Elerp(s_i, s_{i+1}, n), 2n(1-n)) \quad (29)$$

$$R_{i-1}(1) = Esquad(q_{i-1}, q_i, R_{i-1}, R_i, 1) = Elerp(Elerp(q_{i-1}, q_i, 1), Elerp(R_{i-1}, R_i, 1), 0) = q_i$$

$$R_i(0) = Esquad(q_i, q_{i+1}, R_i, R_{i+1}, 0)$$

$$= Elerp(Elerp(q_i, q_{i+1}, 0), Elerp(R_i, R_{i+1}, 0), 0) = q_i$$

$$R_{i-1}(1) = q_i = R_i(0)$$

is displayed. Thus Esquad is continuous and has an accurate value at checkpoints. To match derivatives of two successive spline segments, endpoints to obtain continuous derivatives

$$R'_{i-1}(1) = R'_i(0) \quad (30)$$

We will now show that the Esquad is not consistently differentiable at controlpoints.

$$Elerp(Elerp(q_i, q_{i+1}, n), Elerp(s_i, s_{i+1}, n), 2n(1-n))$$

$$Elerp(q_i, q_{i+1}, n)\begin{pmatrix} Elerp(q_i, q_{i+1}, n)^{-1} \\ Elerp(s_i, s_{i+1}, n) \end{pmatrix}^{2n(1-n)}$$

$$g_i(n) = Elerp(q_i, q_{i+1}, n)^{-1} Elerp(s_i, s_{i+1}, n) \quad (31)$$

Here $g_i(n)$ is the unit elliptic quaternion, so $g_i(n)$ can be written as,

$$g_i(n) = [\cos(\varphi_{g_i(n)}), \sin(\varphi_{g_i(n)}) m_{g_i(n)}] \quad (32)$$

We will now demonstrate that Esquad is continuously differentiable at controlpoints.

$$R_i(n) = Esquad(q_i, q_{i+1}, s_i, s_{i+1}, n) \quad (33)$$

$$R'_i(n) = Esquad(q_i, q_{i+1}, s_i, s_{i+1})$$

$$= \frac{d}{dn} Elerp(Elerp(q_i, q_{i+1}, n), Elerp(s_i, s_{i+1}, n), 2n(1-n)) = \frac{d}{dn}\left(El\ erp(q_i, q_{i+1}, n) g_i(n)^{2n(1-n)}\right)$$

$$= \left(\frac{d}{dn}\left(Elerp(q_i, q_{i+1}, n)\right)\right) g_i(n)^{2n(1-n)} + Elerp(q_i, q_{i+1}, n)\left(\frac{d}{dn} g_i(n)^{2n(1-n)}\right) \quad (34)$$

Using equation (32)

$$\frac{d}{dn}g_i(n)^{2n(1-n)} = \begin{bmatrix} sin\big(2n(1-n)\varphi_{g_i(n)}\big)\Big((2-4n)\varphi_{g_i(n)} + 2n(1-n)\varphi_{g_i'(n)}\Big), \\ cos\big(2n(1-n)\varphi_{g_i(n)}\big)\Big((2-4n)\varphi_{g_i(n)} + 2n(1-n)\varphi_{g_i'(n)}\Big)\, m_{g_i(n)} \\ + sin\big(2n(1-n)\varphi_{g_i(n)}\big)\, m_{g_i'(n)} \end{bmatrix}$$

$$\frac{d}{dn}g_{i-1}(n)^{2n(1-n)}|n=1 = \begin{bmatrix} sin\big(2\cdot 1(1-1)\varphi_{g_{i-1}(1)}\big)\Big((2-4\cdot 1)\varphi_{g_{i-1}(1)} + 2\cdot 1(1-1)\varphi_{g_{i-1}'(1)}\Big), \\ cos\big(2\cdot 1(1-1)\varphi_{g_{i-1}(1)}\big)\Big((2-4\cdot 1)\varphi_{g_{i-1}(1)} + 2\cdot 1(1-1)\varphi_{g_{i-1}'(1)}\Big)\, m_{g_i(n)} \\ + sin\big(2\cdot 1(1-1)\varphi_{g_{i-1}(1)}\big)\, m_{g_{i-1}'(1)} \end{bmatrix}$$

$$\frac{d}{dn}g_{i-1}(n)^{2n(1-n)}|n=1 = [0, -2\phi_{g_{i-1}}(1)\varepsilon_{g_{i-1}}(1)]$$

$$\frac{d}{dn}g_i(n)^{2n(1-n)}|n=0 = \begin{bmatrix} sin\big(2\cdot 0(1-0)\varphi_{g_i(0)}\big)\Big((2-4\cdot 0)\varphi_{g_i(0)} + 2\cdot 0(1-0)\varphi_{g_i'(0)}\Big), \\ cos\big(2\cdot 0(1-0)\varphi_{g_i(0)}\big)\Big((2-4\cdot 0)\varphi_{g_i(0)} + 2\cdot 0(1-0)\varphi_{g_i'(0)}\Big)\, m_{g_i(0)} \\ + sin\big(2\cdot 0(1-0)\varphi_{g_i(0)}\big)\, m_{g_i'(0)} \end{bmatrix}$$

$$\frac{d}{dn}g_i(n)^{2n(1-n)}|n=0 = [0, 2\varphi_{g_i}(0)m_{g_i}(0)]$$

using equation (34), we get:

$$\begin{aligned} &\frac{d}{dt}Esquad(q_{i-1}, q_i, s_{i-1}, s_i, 1) = \\ &Elerp(q_{i-1}, q_i, 1)\, log(q_{i-1}^{-1}, q_i) \\ &+ Elerp(q_{i-1}, q_i, 1)\left(\frac{d}{dn}g_i(n)^{2n(1-n)}|n=1\right) = \\ &q_i\, log(q_{i-1}^{-1}, q_i) + q_i\big[0, -2\varphi_{i-1}(1)m_{g_{i-1}}(1)\big] = \\ &q_i\Big(log(q_{i-1}^{-1}, q_i) - \\ &2\, log\Big(\Big[cos\big(\varphi_{g_{i-1}}(1)\big), sin\big(\varphi_{g_{i-1}}(1)\big)\, m_{g_{i-1}}(1)\Big]\Big)\Big) = \\ &q_i\big(log(q_{i-1}^{-1}, q_i) - 2\, log\big(g_{i-1}(1)\big)\big) = \\ &q_i\big(log(q_{i-1}^{-1}, q_i) - 2\, log(q_i^{-1}s_i)\big) \end{aligned} \tag{35}$$

$$\begin{aligned} &\frac{d}{dt}Esquad(q_i, q_{i+1}, s_i, s_{i+1}, 0) = \\ &Elerp(q_i, q_{i+1}, 0)\, log(q_i^{-1}, q_{i+1}) + \\ &Elerp(q_i, q_{i+1}, 0)\left(\frac{d}{dn}g_i(n)^{2n(1-n)}|n=0\right) = \\ &q_i\, log(q_i^{-1}, q_{i+1}) + q_i\big[0, 2\varphi_i(0)m_{g_i}(0)\big] = \\ &q_i\Big(log(q_i^{-1}, q_{i+1}) + \\ &2\, log\Big(\Big[cos\big(\varphi_{g_i}(0)\big), sin\big(\varphi_{g_i}(0)\big)\, m_{g_i}(0)\Big]\Big)\Big) = \\ &q_i\big(log(q_i^{-1}, q_{i+1}) + 2\, log\big(g_i(0)\big)\big) = \\ &q_i\big(log(q_i^{-1}, q_{i+1}) + 2\, log(q_i^{-1}s_i)\big) \end{aligned} \tag{36}$$

using equation (35) and (36) s_i must satisfy

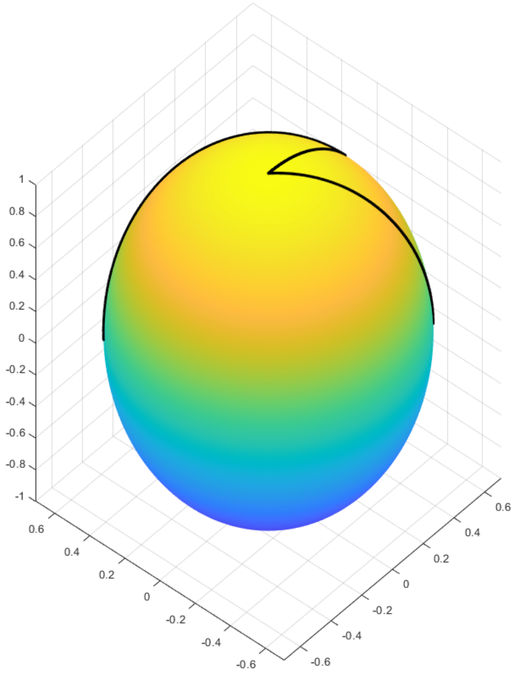
$$q_i\, log\big((q_{i-1}^{-1}, q_i) - 2\, log(q_i^{-1}s_i)\big) = q_i\, log\big((q_i^{-1}, q_{i+1}) + 2\, log(q_i^{-1}s_i)\big)$$

$$s_i = q_i\, exp\left(-\frac{log(q_i^{-1}q_{i-1}) + log(q_i^{-1}q_{i+1})}{4}\right)$$

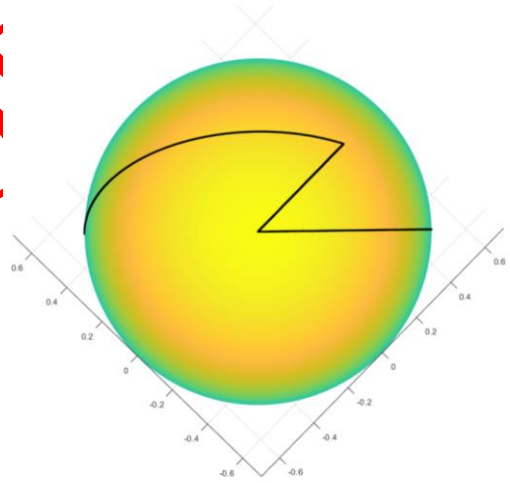
s_i is obtained.

As a result, using the above definition of s_i, Esquad is continuously differentiable at control points. In reality, we've demonstrated that it differs throughout every Esquad section consistently and continually.

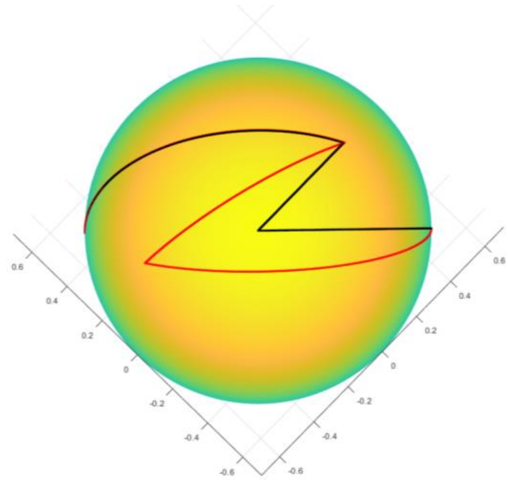
(see figure 3).



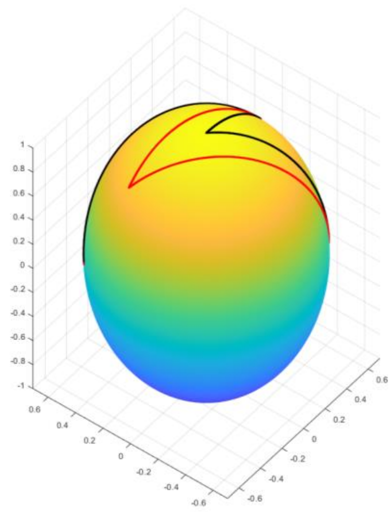
(a)



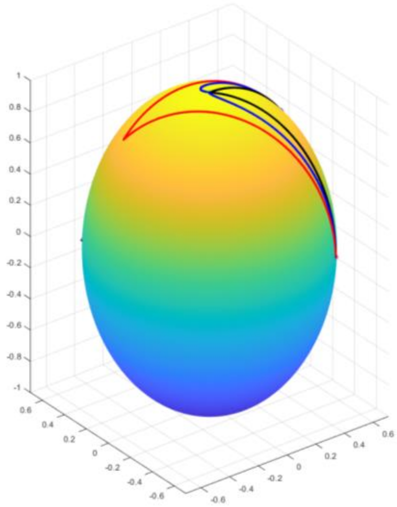
(b)



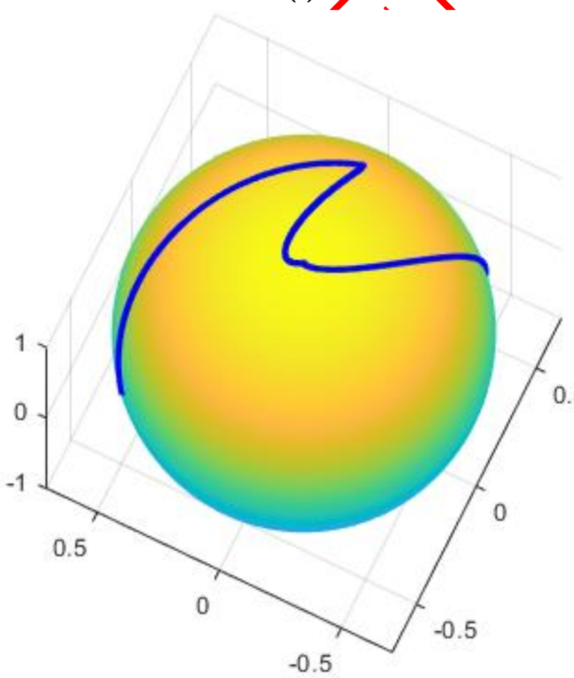
(c)



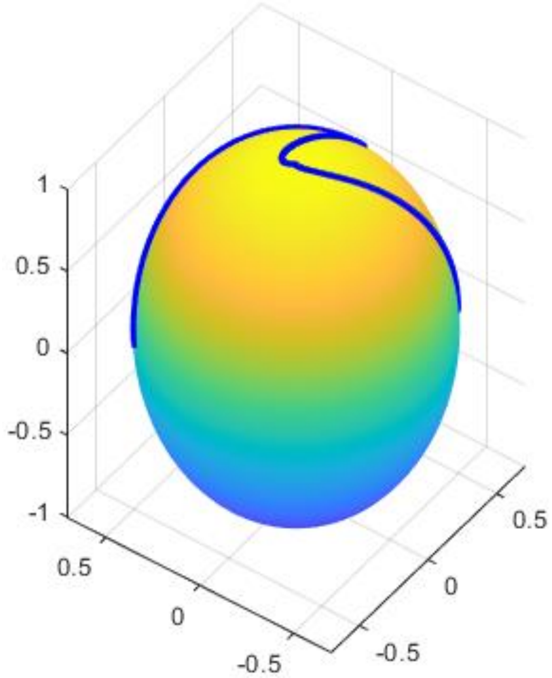
(d)



(e)



(f)



(g)

Figure 3. (a) elliptic quaternion interpolation between the four key frames on ellipsoid; (b) Inner quadrangle interpolation between the four key frames on ellipsoid; (c) Combination of elliptic quaternion and inner quadrangle on the ellipsoid; (d) other view Combination of elliptic quaternion and inner quadrangle on ellipsoid; (e) Smoothing elliptic quaternion with using inner quadrangle; (f) interpolation curve for ESquad; (g) other view interpolation curve for ESquad

6. CONCLUSION

We presented the spline elliptic quaternion interpolation on the ellipsoid in this paper. On the ellipsoid, elliptic quaternions have a group structure. The ESquad (spline elliptic quaternion interpolation) is defined on the ellipsoid using these features. The suggested technique consists of building a cubic interpolation out of three linear interpolations. For a collection of elliptic quaternions $\{q_n\}_{n=0}^{N-1}$, the algorithmic expression for ESquad yields an interpolation curve. We have demonstrated how consistently and continually differentiable ESquad is across all segment. In other words, we seamlessly interpolated between a series of position and direction interpolations on the ellipsoid.

DECLARATION OF ETHICAL STANDARDS

This article does not contain any studies with human participants or animals performed by any of the authors.

AUTHORS' CONTRIBUTIONS

Raheleh GHADAMI: Performed the experiments, analyse the results and wrote manuscript.

Yusuf YAYLI: Performed the experiments, analyse the results and wrote manuscript.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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