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Quasi-Lacunary Invariant Statistical Convergence of Sequences of Sets

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Abstract

In this study, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman quasi-strongly lacunary invariant convergence and Wijsman quasi-strongly q-lacunary invariant convergence for sequences of sets. Also we define Wijsman quasi-lacunary invariant statistical convergence. Then, we examine the existence of the relations among these new convergence types and some convergence types for sequences of sets given before. Furthermore, we examine the existence of the relations between some of these new convergence types, too.

Keywords: *Quasi-invariant convergence, lacunary sequence, statistical convergence, Wijsman convergence, sequences of sets.*
2010 Mathematics Subject Classification: *40A05, 40A35*

1. Introduction and Background

The concept of statistical convergence was firstly introduced by Fast [6] and this concept has been studied by Šalát [25], Fridy [7], Connor [4] and many others [5, 23, 26, 27, 31, 32], too.
A sequence $x=(x_k)$ is statistically convergent to L if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\Big|\big\{k\le n:|x_k-L|\ge\varepsilon\big\}\Big|=0,$$

where the vertical bars indicate the number of elements in the enclosed set.
By a lacunary sequence we mean an increasing integer sequence $\theta=\{k_r\}$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. Throughout this study, the intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$.
Then, using the lacunary sequence concept, Fridy and Orhan [8] defined the concept of lacunary statistical convergence.
Let $\theta=\{k_r\}$ be a lacunary sequence. A sequence $x=(x_k)$ is lacunary statistically convergent to L if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\big\{k\in I_r:|x_k-L|\ge\varepsilon\big\}\Big|=0.$$

Many authors have studied on the concepts of invariant mean and invariant convergence [11, 12, 13, 19, 20, 24].
Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1) $\phi(x)\ge0$, when the sequence $x=(x_n)$ has $x_n\ge0$ for all n,
2) $\phi(e)=1$, where $e=(1,1,1,...)$ and
3) $\phi(x_{\sigma(n)})=\phi(x_n)$ for all $x\in\ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n)\neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x)=\lim x$ for all $x\in c$.
In the case σ is translation mappings $\sigma(n)=n+1$, the σ-mean is often called a Banach limit.
The space of lacunary strong σ-convergent sequences L_θ was introduced by Savaş [21] as below:

$$L_\theta=\left\{x=(x_k):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|x_{\sigma^k(m)}-L|=0,\ \text{uniformly in } m\right\}.$$

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Then, Savaş and Nuray [22] defined the concept of lacunary σ-statistically convergent sequence.
Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is $S_{\sigma\theta}$-convergent to L if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, \ldots$.
Recently, the concepts of lacunary invariant summability and strongly lacunary q-invariant convergence were studied by Pancaroğlu and Nuray [17].
A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(n)} = L$$

uniformly in $n = 1, 2, \ldots$.
Let $0 < q < \infty$. A sequence $x = (x_k)$ is said to be strongly lacunary q-invariant convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(n)} - L|^q = 0$$

uniformly in n and it is denoted by $x_k \to L\big([V_{\sigma\theta}]_q\big)$.
Let X be any non-empty set and $\mathbb{N}$ be the set of natural numbers. The function

$$f : \mathbb{N} \to P(X)$$

is defined by $f(k) = A_k \in P(X)$ for each $k \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{A_k\} = (A_1, A_2, \ldots)$, which is the range's elements of f, is said to be sequences of sets.
Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, the distance from x to A is defined by

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

Throughout this study, we will take (X, ρ) as a metric space and A, A_k as any non-empty closed subsets of X.
There are different convergence concepts for sequences of sets. One of them handled in this paper is the concept of Wijsman convergence [1, 2, 3, 14, 18, 30, 33, 34].
A sequence $\{A_k\}$ is said to be Wijsman convergent to A if for each $x \in X$,

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

and denoted by $A_k \xrightarrow{W} A$.
A sequence $\{A_k\}$ is said to be bounded if for each $x \in X$, there exists an $M > 0$ such that $|d(x, A_k)| < M$ for all k, i.e., $\sup_k \{d(x, A_k)\} < \infty$. The set of all bounded sequences of sets is denoted by L_∞.
The concepts of Wijsman lacunary summability, Wijsman strongly lacunary summability and Wijsman lacunary statistical convergence were introduced by Ulusu and Nuray [28, 29].
Then, using the invariant mean concept, Pancaroğlu and Nuray [16] defined the concepts of Wijsman lacunary invariant convergence, Wijsman strongly lacunary invariant convergence and Wijsman lacunary invariant statistical convergence.
Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_{\sigma^k(m)}) = d(x, A)$$

uniformly in m.
A sequence $\{A_k\}$ is said to be Wijsman strongly lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| d(x, A_{\sigma^k(m)}) - d(x, A) \right| = 0$$

uniformly in m.
A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant statistically convergent to A if for each $x \in X$ and every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon \right\} \right| = 0$$

uniformly in m.
The idea of quasi-almost convergence in a normed space was introduced by Hajduković [10]. Then, Nuray [15] studied the concepts of quasi-invariant convergence and quasi-invariant statistical convergence in a normed space.
Recently, the concepts of Wijsman quasi-strongly invariant convergence and Wijsman quasi-invariant statistical convergence for sequences of sets were introduced by Gülle and Ulusu [9] as below:
A sequence $\{A_k\}$ is said to be Wijsman quasi-strongly invariant convergent to A if for each $x \in X$,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} \left| d_x(A_{\sigma^k(np)}) - d_x(A) \right| = 0$$

uniformly in n, where $d_x(A_{\sigma^k(np)}) = d(x, A_{\sigma^k(np)})$ and $d_x(A) = d(x,A)$. It is denoted by $A_k \xrightarrow{[WQV_\sigma]} A$.
A sequence $\{A_k\}$ is said to be Wijsman quasi-invariant statistically convergent to A if for each $x \in X$ and every $\varepsilon > 0$,

$$\lim_{p\to\infty} \frac{1}{p} \left| \left\{ k \leq p : |d_x(A_{\sigma^k(np)}) - d_x(A)| \geq \varepsilon \right\} \right| = 0$$

uniformly in n. It is denoted by $A_k \xrightarrow{WQS_\sigma} A$.

2. Main Results

In this section, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman quasi-strongly lacunary invariant convergence and Wijsman quasi-strongly q-lacunary invariant convergence for sequences of sets. Also we define Wijsman quasi-lacunary invariant statistical convergence. Then, we examine the existence of the relations among these new convergence types and some convergence types for sequences of sets given before. Furthermore, we examine the existence of the relations between some of these new convergence types, too.

Definition 2.1. *Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A if for each $x \in X$,*

$$\lim_{r\to\infty} \left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| = 0$$

uniformly in n. In this case, we write $A_k \xrightarrow{WQV_{\sigma\theta}} A$.

Theorem 2.2. *If a sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A, then the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A. Then, for each $x \in X$ and every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(m)}) - d_x(A) \right| < \varepsilon,$$

for all m. If m is taken as $m = nr$, then we get

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| < \varepsilon,$$

for all n. Since $\varepsilon > 0$ is an arbitrary, if the limit is taken for $r \to \infty$ we have

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| \to 0$$

for all n. Thus, the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A. □

Definition 2.3. *Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman quasi-strongly lacunary invariant convergent to A if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| d_x(A_{\sigma^k(nr)}) - d_x(A) \right| = 0$$

uniformly in n. In this case, we write $A_k \xrightarrow{[WQV_{\sigma\theta}]} A$.

Theorem 2.4. *For any lacunary sequence $\theta = \{k_r\}$,*

$$A_k \xrightarrow{[WQV_{\sigma\theta}]} A \Leftrightarrow A_k \xrightarrow{[WQV_\sigma]} A.$$

Proof. Let $A_k \xrightarrow{[WQV_{\sigma\theta}]} A$ and $\varepsilon > 0$ is given. Then, there exists an integer r_0 such that for each $x \in X$

$$\frac{1}{h_r} \sum_{k=0}^{h_r-1} \left| d_x(A_{\sigma^k(nr)}) - d_x(A) \right| < \varepsilon,$$

for $r \geq r_0$ and $nr = k_{r-1} + 1 + w$, $w \geq 0$. Let $p \geq h_r$. Thus, p can be written as $p = \alpha h_r + t$ where α is an integer and $0 \leq t \leq h_r$. Since $p \geq h_r$, $\alpha \geq 1$. Then, we get

$$\begin{aligned}
\frac{1}{p} \sum_{k=0}^{p-1} \left| d_x(A_{\sigma^k(np)}) - d_x(A) \right| &\leq \frac{1}{p} \sum_{k=0}^{(\alpha+1)h_r-1} \left| d_x(A_{\sigma^k(nr)}) - d_x(A) \right| \\
&= \frac{1}{p} \sum_{j=0}^{\alpha} \sum_{k=jh_r}^{(j+1)h_r-1} \left| d_x(A_{\sigma^k(nr)}) - d_x(A) \right| \\
&\leq \frac{1}{p} \varepsilon h_r (\alpha+1) \\
&\leq \frac{2\alpha h_r \varepsilon}{p} \quad (\alpha \geq 1).
\end{aligned}$$

Since $\dfrac{\alpha h_r}{p} \leq 1$, we have

$$\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right| \leq 2\varepsilon,$$

therefore $A_k \overset{[WQV_\sigma]}{\longrightarrow} A$.

Let $A_k \overset{[WQV_\sigma]}{\longrightarrow} A$ and $\varepsilon > 0$ is given. Then, there exists a number $P > 0$ such that for each $x \in X$,

$$\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right| < \varepsilon$$

for all $p > P$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > P$ where $r \geq R$. Thereby, we get

$$\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| < \varepsilon.$$

This implies that $A_k \overset{[WQV_{\sigma\theta}]}{\longrightarrow} A$. □

Definition 2.5. *Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A if for each $x \in X$ and every $\varepsilon > 0$,*

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| = 0$$

uniformly in n. In this case, we write $A_k \overset{WQS_{\sigma\theta}}{\longrightarrow} A$.

The set of all Wijsman quasi-lacunary invariant statistically convergent sequences will be denoted by $(WQS_{\sigma\theta})$.

Theorem 2.6. *If a sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A, then the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A. In this case, when $\delta > 0$ is given, for each $x \in X$ and every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(m)}) - d_x(A)| \geq \varepsilon\right\}\right| < \delta,$$

for all m. If m is taken as $m = nr$, then we get

$$\frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| < \delta,$$

for all n. Since $\delta > 0$ is an arbitrary, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| = 0,$$

uniformly in n which means that the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A. □

Theorem 2.7. *For any lacunary sequence $\theta = \{k_r\}$,*

$$A_k \overset{WQS_{\sigma\theta}}{\longrightarrow} A \Leftrightarrow A_k \overset{WQS_\sigma}{\longrightarrow} A.$$

Proof. Let $A_k \overset{WQS_{\sigma\theta}}{\longrightarrow} A$ and $\delta > 0$ is given. Then, there exists an integer r_0 such that for each $x \in X$

$$\frac{1}{h_r}\left|\left\{0 \leq k \leq h_r - 1 : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| \leq \delta,$$

for $r \geq r_0$ and $nr = k_{r-1} + 1 + w$, $w \geq 0$. Let $p \geq h_r$. Thus, p can be written as $p = \alpha h_r + t$ where α is an integer and $0 \leq t \leq h_r$. Since $p \geq h_r$, $\alpha \geq 1$. Then, we get

$$\begin{aligned}
\frac{1}{p}\left|\left\{k \leq p-1 : |d_x(A_{\sigma^k(np)}) - d_x(A)| \geq \varepsilon\right\}\right| &\leq \frac{1}{p}\left|\left\{k \leq (\alpha+1)h_r - 1 : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| \\
&= \frac{1}{p}\sum_{j=0}^{\alpha}\left|\left\{jh_r \leq k \leq (j+1)h_r - 1 : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{p}\,\delta\, h_r\,(\alpha+1) \\
&\leq \frac{2\alpha\, h_r\,\delta}{p}.
\end{aligned}$$

Since $\dfrac{\alpha h_r}{p} \le 1$, we have

$$\frac{1}{p}\left|\left\{0 \le k \le p-1 : |d_x(A_{\sigma^k(np)}) - d_x(A)| \ge \varepsilon\right\}\right| \le 2\delta,$$

therefore $A_k \overset{WQS_\sigma}{\longrightarrow} A$.

Let $A_k \overset{WQS_\sigma}{\longrightarrow} A$ and $\zeta > 0$ is given. Then, there exists a number $P > 0$ such that for each $x \in X$,

$$\frac{1}{p}\left|\left\{k \le p : |d_x(A_{\sigma^k(np)}) - d_x(A)| \ge \varepsilon\right\}\right| < \zeta$$

for all $p > P$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > P$ where $r \ge R$. Thereby, we get

$$\frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon\right\}\right| < \zeta.$$

This implies that $A_k \overset{WQS_{\sigma\theta}}{\longrightarrow} A$. □

Definition 2.8. *Let $\theta = \{k_r\}$ be a lacunary sequence and $0 < q < \infty$. A sequence $\{A_k\}$ is Wijsman quasi-strongly q-lacunary invariant convergent to A if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q = 0$$

uniformly in n. In this case, we write $A_k \overset{[WQV_{\sigma\theta}]^q}{\longrightarrow} A$.

The set of all Wijsman quasi-strongly q-lacunary invariant convergent sequences will be denoted by $[WQV_{\sigma\theta}]^q$.

Theorem 2.9. *If a sequence $\{A_k\}$ is Wijsman quasi-strongly q-lacunary invariant convergent to A, then this sequence is Wijsman quasi-lacunary invariant statistically convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman quasi-strongly q-lacunary invariant convergent to A. Then, for each $x \in X$ and every $\varepsilon > 0$ following inequality is provided:

$$\begin{aligned}
\sum_{k \in I_r} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q &= \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon}} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q + \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(nr)}) - d_x(A)| < \varepsilon}} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \\
&\ge \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon}} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \\
&\ge \varepsilon^q \left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon\right\}\right|,
\end{aligned}$$

that is,

$$\sum_{k \in I_r} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \ge \varepsilon^q \left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon\right\}\right| \tag{2.1}$$

for all n. If the both side of the inequality (2.1) are multiplied by $\dfrac{1}{h_r}$ and after that the limit is taken for $r \to \infty$, due to our acceptance, we get

$$0 = \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \ge \varepsilon^q \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon\right\}\right|.$$

Hence, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \ge \varepsilon\right\}\right| = 0$$

uniformly in n, that is, $A_k \overset{WQS_{\sigma\theta}}{\longrightarrow} A$. □

Theorem 2.10. *If a sequence $\{A_k\} \in L_\infty$ and Wijsman quasi-lacunary invariant statistically convergent to A, then this sequence is Wijsman quasi-strongly q-lacunary invariant convergent to A.*

Proof. Suppose that the sequence $\{A_k\} \in L_\infty$ and Wijsman quasi-lacunary invariant statistically convergent to A. Since $\{A_k\}$ is bounded, there exists an $M > 0$ such that for each $x \in X$,

$$\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right|^q \leq M.$$

Then, for each $x \in X$ and every $\varepsilon > 0$ we can write

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q &= \frac{1}{h_r}\left(\sum_{\substack{k\in I_r\\ |d_x(A_{\sigma^k(nr)})-d_x(A)|\geq\varepsilon}}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q + \sum_{\substack{k\in I_r\\ |d_x(A_{\sigma^k(nr)})-d_x(A)|<\varepsilon}}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q\right)\\
&\leq \frac{M}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| + \frac{\varepsilon^q}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| < \varepsilon\right\}\right|\\
&= \frac{M}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| + \varepsilon^q,
\end{aligned}$$

that is,

$$\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \leq \frac{M}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| + \varepsilon^q \tag{2.2}$$

for all n. If the limit of both side of the inequality (2.2) is taken for $r\to\infty$, due to our acceptance, we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q \leq \lim_{r\to\infty}\left(\frac{M}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| + \varepsilon^q\right) = \varepsilon^q.$$

Thus, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right|^q = 0$$

uniformly in n, that is, $A_k \stackrel{[WQV_{\sigma\theta}]^q}{\longrightarrow} A$. □

From Theorem 2.9 and Theorem 2.10, we have following corollary.

Corollary 2.11. $(WQS_{\sigma\theta}) \cap L_\infty = [WQV_{\sigma\theta}]^q$.

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